ETH zürich

Trigonometrie

Die Additionstheoreme

Meike Akveld

Motivation

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Satz (Additionstheoreme)

$$\sin(\alpha + \beta) = \sin(\alpha) \cdot \cos(\beta) + \cos(\alpha) \cdot \sin(\beta)$$

$$\cos(\alpha + \beta) = \cos(\alpha) \cdot \cos(\beta) - \sin(\alpha) \cdot \sin(\beta)$$

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Beispiel:

$$\sin\left(\frac{5\pi}{12}\right) = \sin\left(\underbrace{\frac{\pi}{4}}_{\alpha} + \underbrace{\frac{\pi}{6}}_{\beta}\right) = \sin\left(\frac{\pi}{4}\right) \cdot \cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{4}\right) \cdot \sin\left(\frac{\pi}{6}\right)$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{2} + \sqrt{6}}{4}$$

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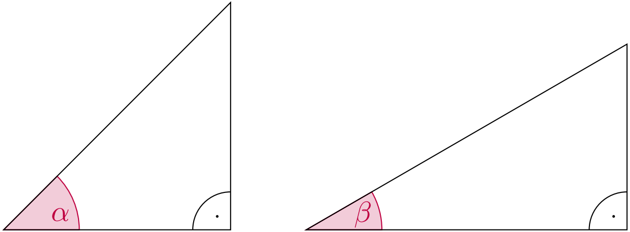
Beweis für den Fall $0 < \alpha, \beta, \alpha + \beta < 90^\circ$



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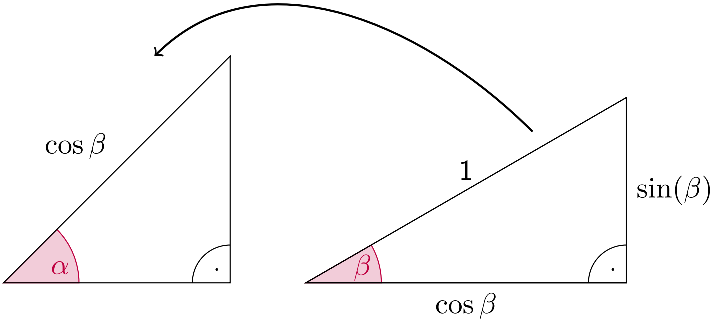

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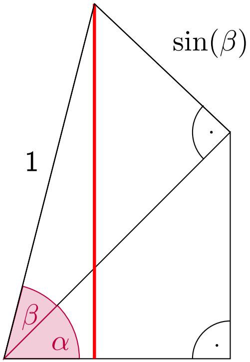

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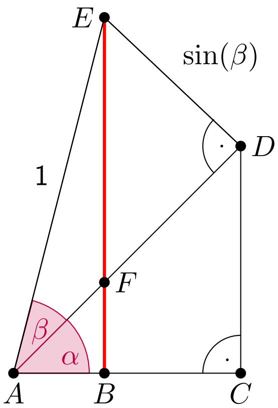
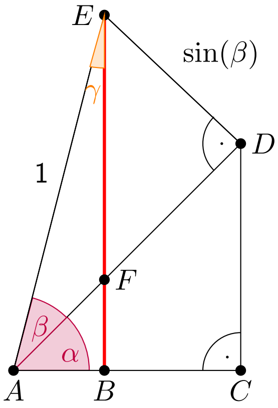


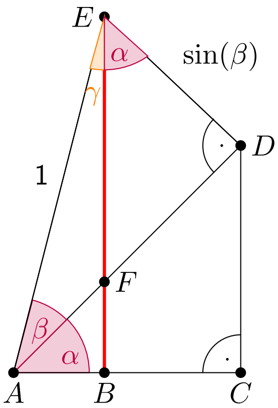
$\overline{EB} = \sin(\alpha + \beta)$ und $\overline{EB} = \overline{EF} + \overline{FB}$

Beweis für den Fall $0 < \alpha, \beta, \alpha + \beta < 90^\circ$



Im Dreieck ΔABE gilt $\gamma = \angle AEB = 90^\circ - (\alpha + \beta)$

Beweis für den Fall $0 < \alpha, \beta, \alpha + \beta < 90^\circ$

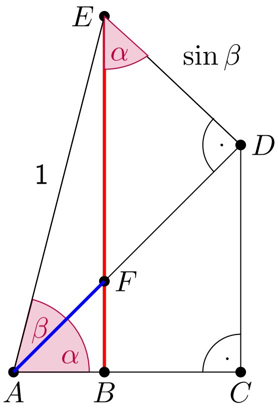


Im Dreieck ΔABE gilt $\gamma = \angle AEB = 90^\circ - (\alpha + \beta)$

Im Dreieck ΔADE gilt $\angle AED = 90^\circ - \beta$ und somit

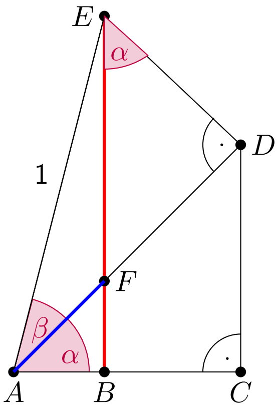
$\angle FED = \angle AED - \gamma = 90^\circ - \beta - (90^\circ - (\alpha + \beta)) = \alpha$

Beweis für den Fall $0 < \alpha, \beta, \alpha + \beta < 90^\circ$



Also $\dfrac{\overline{DE}}{\overline{EF}} = \cos(\alpha) \implies \overline{EF} = \dfrac{\overline{DE}}{\cos(\alpha)} = \dfrac{\sin(\beta)}{\cos(\alpha)}$

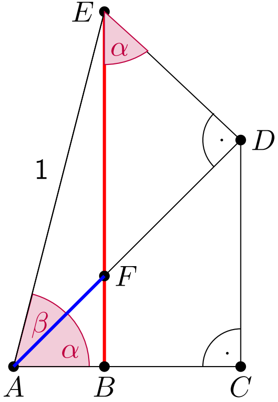
Beweis für den Fall $0 < \alpha, \beta, \alpha + \beta < 90^\circ$



Anderseits gilt $\overline{AF} = \overline{AD} - \overline{FD}$ und da $\overline{AD} = \cos(\beta)$ und $\overline{FD} = \overline{DE} \cdot \tan(\alpha)$, haben wir

$$\overline{AF} = \cos(\beta) - \overline{DE} \cdot \tan(\alpha) = \cos(\beta) - \sin(\beta) \cdot \frac{\sin(\alpha)}{\cos(\alpha)}$$

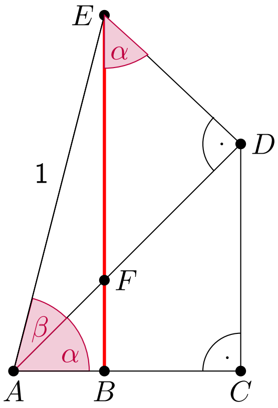
Beweis für den Fall $0 < \alpha, \beta, \alpha + \beta < 90^\circ$



Es folgt $\dfrac{\overline{BF}}{\overline{AF}} = \sin(\alpha)$ und somit

$$\overline{BF} = \overline{AF} \cdot \sin(\alpha) = \cos(\beta) \cdot \sin(\alpha) - \frac{\sin(\beta)\sin^2(\alpha)}{\cos(\alpha)}$$

Beweis für den Fall $0 < \alpha, \beta, \alpha + \beta < 90^\circ$



Da nun $\sin(\alpha+\beta) = \overline{EF} + \overline{BF}$ erhalten wir durch einsetzen

$$\sin(\alpha+\beta) = \frac{\sin(\beta)}{\cos(\alpha)} + \cos(\beta)\cdot\sin(\alpha) - \frac{\sin(\beta)\sin^2(\alpha)}{\cos(\alpha)}$$

$$= \cos(\beta)\cdot\sin(\alpha) + \sin(\beta)\cdot\frac{1-\sin^2(\alpha)}{\cos(\alpha)}$$

$$= \cos(\beta)\cdot\sin(\alpha) + \sin(\beta)\cdot\cos(\alpha)$$

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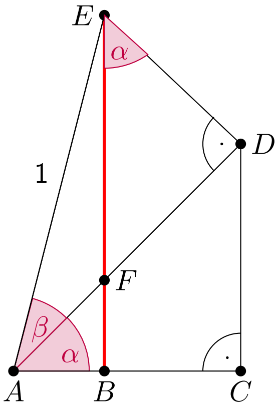

Satz (Additionstheorem für Sinus)

$$\sin(\alpha + \beta) = \sin(\alpha) \cdot \cos(\beta) + \cos(\alpha) \cdot \sin(\beta)$$

Eine erste Anwendung

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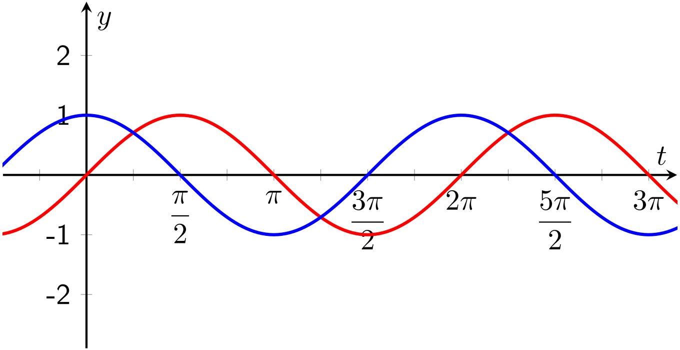

$f(t) = \sin(t)$

$g(t) = \sin\left(t + \frac{\pi}{2}\right)$

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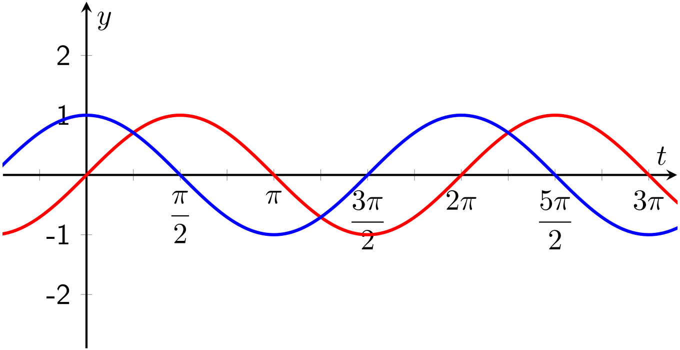

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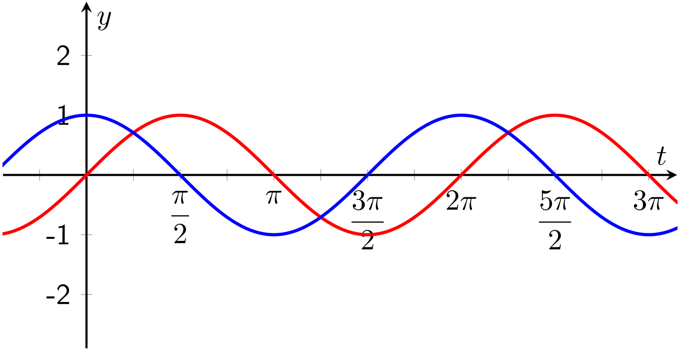

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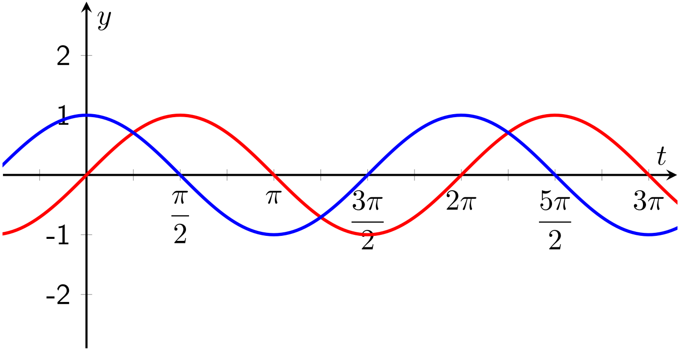

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